

Maximum Loss of Certain Lévy Processes¹

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Abstract

The distribution of maximum loss is considered for a stable Lévy process and a spectrally negative Lévy process. The tail behavior is found for the stable process, while the latter is taken up to the passage time of a given level, or an independent exponential time. Path decomposition is performed to analyze the pre-supremum and the post-infimum processes as motivated by the aim of finding the joint distribution of the maximum loss and maximum gain.

Keywords: maximum drawdown, stable, spectrally negative, reflected process, fluctuation theory

1. Introduction

The maximum loss, or maximum drawdown of a process X is the supremum of X reflected at its running supremum. The motivation for the loss terminology comes from mathematical finance as it is useful to quantify the risk associated with the performance of a stock. The loss process, or the so-called drawdown, has been studied in particular for Brownian motion (Salminen and Valois, 2007, Vardar and Zirbel, 2013), and some Lévy processes (Mijatovic and Pistorius, 2012).

Maximum loss at time $t > 0$ is formally defined by

$$M_t^- := \sup_{0 \leq u \leq v \leq t} (X_u - X_v),$$

which is equivalent to $\sup_{0 \leq v \leq t} (\sup_{0 \leq u \leq v} (X_u - X_v))$ and $\sup_{0 \leq v \leq t} (S_v - X_v)$, that is, the supremum of the reflected process $S - X$, or the so-called loss process,

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where S denotes the running supremum. Similarly, a counterpart quantity called the maximum gain is given by

$$M_t^+ := \sup_{0 \leq u \leq v \leq t} (X_v - X_u) = \sup_{0 \leq v \leq t} (X_v - I_v)$$

where I is the running infimum. It is easy to see that maximum gain of X is the maximum loss of the dual process $-X$. The gain process is the dual process $-X$ reflected at its supremum, that is, $-I - (-X) = X - I$.

In this paper, we consider the distribution of the maximum loss of stable Lévy processes and spectrally negative Lévy processes. We indicate the analogous results for the maximum gain where applicable, and consider their joint distributions when possible. An α -stable Lévy motion is a Lévy process that satisfies the self-similarity property with index $\alpha \in (0, 2]$, namely, the probability laws of $\{X_t : t \geq 0\}$ and $\{b^{-1/\alpha} X_{bt} : t \geq 0\}$ are the same for every scaling parameter $b > 0$. When $0 < \alpha < 1$ or $1 < \alpha < 2$, its characteristic exponent is given for λ by

$$\Psi(\lambda) = c|\lambda|^\alpha (1 - i\beta \operatorname{sgn}(\lambda) \tan(\pi\alpha/2)) \quad (1)$$

where $\beta \in [-1, 1]$ and $c > 0$. In this case, the Lévy measure has the form $|x|^{-\alpha-1} dx$. For the special cases $\alpha = 1$ and $\alpha = 2$, the process is a symmetric Cauchy process with drift and a Brownian motion, respectively (see e.g., Bertoin, 2007). On the other hand, a spectrally negative Lévy process X is a Lévy process with no positive jumps, that is, its Lévy measure Π is concentrated on $(-\infty, 0)$. Let ψ denote the Laplace exponent of X , as found by $\mathbb{E}(\exp \lambda X_t) = \exp(t\psi(\lambda))$, which is finite for all $\lambda > 0$. It is related to the characteristic exponent Ψ by $\psi(\lambda) = -\Psi(-i\lambda)$. The Laplace exponent of a spectrally negative Lévy process is given by

$$\psi(\lambda) = -\mu\lambda + \frac{\sigma^2}{2}\lambda^2 + \int_{(-\infty, 0)} (e^{\lambda x} - 1 - \lambda x 1_{\{x > -1\}}) \Pi(dx)$$

where $\mu \in \mathbb{R}$, and $\sigma > 0$. Note that Brownian motion with drift μ can be taken as a special case when $\Pi \equiv 0$.

We first find the tail behavior of the distribution of maximum loss for a stable Lévy process X , which turns out to be a power-like decay given by

$$\mathbb{P}(M_t^- > x) \sim ktx^{-\alpha}$$

for some constant $k > 0$, and $t \in \mathbb{R}_+$, as $x \rightarrow \infty$, in Theorem 1. The tail behavior is similar to that for the marginal distribution and the supremum. It is easy to see that maximum gain M_t^+ also shows the same tail behavior.

We also consider a spectrally negative Lévy process X excluding the trivial cases of the negative of a subordinator or a deterministic drift. The distribution of the maximum loss up to the passage time of the process above a level is given in Theorem 2 as a particular case of joint distribution of the maximum loss and the two-sided exit. The joint distribution of maximum loss and maximum gain appears as a corollary. The result is given in terms of the so-called scale function W of X , which is defined as the inverse Laplace transform of $1/\psi(\lambda)$, and the excursion measure of $S - X$. For an extensive survey of scale functions for spectrally negative Lévy processes, see Kuznetsov et al. (2013). By drawing upon the existing results, we observe the marginal distributions of maximum loss and maximum gain up to an independent exponential time T . Moreover, in search for the joint distribution of M_t^- and M_t^+ , we follow the approach that has been considered previously for Brownian motion without drift in Salminen and Valois (2007). Through path decompositions at the time of the last supremum, and at the time of last infimum before an exponential time, we find the distributions of the pre-supremum process and post-infimum process in Theorem 3. We also find the joint distribution of the infimum and the supremum, which would be useful. However, a thorough path-decomposition including the path between the infimum and the supremum, and hence the joint distribution of M_t^- and M_t^+ remains as future work.

The paper is organized as follows. In Section 2, we consider stable Lévy processes. In Section 3, a spectrally negative Lévy process is studied up to a passage time, as well as before an exponential time. Path decomposition is performed to analyze the pre-supremum and the post-infimum processes in Section 4, and the related open problem is stated.

2. Maximum Loss of Stable Lévy Processes

In this section, we give the asymptotic distribution of maximum loss for a stable Lévy process. Since the dual process $-X$ is also a stable Lévy process with the same tail behavior, the asymptotic distribution of the maximum gain is similar.

Theorem 1. *Suppose X is an α -stable Lévy process which has negative jumps. Then, there exists $k > 0$ such that the tail of the distribution of the maximum loss at $t > 0$ satisfies*

$$\mathbb{P}(M_t^- > x) \sim ktx^{-\alpha}$$

as $x \rightarrow \infty$.

Proof: For all $t > 0$

$$-I_t = \inf_{0 \leq v \leq t} X_v \leq \sup_{0 \leq v \leq t} \sup_{0 \leq u \leq v} (X_u - X_v) = M_t^-$$

It follows that $\mathbb{P}(M_t^- > x) \geq \mathbb{P}(-I_t > x)$. By Bertoin (2007, Prop.VIII.4), there exists $k > 0$ such that $\mathbb{P}\{-I_1 > x\} \sim kx^{-\alpha}$ as $x \rightarrow \infty$ since $-X$ possesses positive jumps. Therefore, we have

$$\liminf_{x \rightarrow \infty} \mathbb{P}(M_1^- > x)x^\alpha \geq \lim_{x \rightarrow \infty} \mathbb{P}(-I_1 > x)x^\alpha = k. \quad (2)$$

On the other hand, the loss process, or drawdown, given by

$$Y_v := \sup_{0 \leq u \leq v} (X_u - X_v)$$

satisfies

$$\mathbb{P}(Y_1 > (1 - \varepsilon)x) \geq \mathbb{P}(\sup_{0 \leq v \leq 1} Y_v > x, Y_1 > (1 - \varepsilon)x) \quad (3)$$

for every $\varepsilon > 0$. For $t \in (0, 1)$, we have

$$Y_1 = \max(Y_t + X_t - X_1, \sup_{t \leq u \leq 1} (X_u - X_1)).$$

Therefore, $\{Y_1 > (1 - \varepsilon)x\} \supset \{Y_t + X_t - X_1 > (1 - \varepsilon)x\}$. Let the passage time of the loss process Y to a level $x > 0$ be $T_x := \inf\{t \geq 0 : Y_t > x\}$ and Γ denote its distribution. For the right hand side of (3), we can write the lower bound

$$\begin{aligned} \mathbb{P}(\sup_{0 \leq v \leq 1} Y_v > x, Y_1 > (1 - \varepsilon)x) &\geq \int_0^1 \mathbb{P}(Y_t + X_t - X_1 > (1 - \varepsilon)x | T_x = t) \Gamma(dt) \\ &= \int_0^1 \mathbb{P}(X_t - X_1 > -\varepsilon x) \Gamma(dt) \end{aligned} \quad (4)$$

by the independence of the increments of X and the fact that $Y_{T_x} \leq x$ as the loss can exceed x either by a negative jump or creeping downwards at T_x . The stationarity of the increments and self-similarity property implies

$$\mathbb{P}(X_t - X_1 > -\varepsilon x) = \mathbb{P}(-X_{1-t} > -\varepsilon x) = \mathbb{P}(-X_1 > -(1-t)^{-1/\alpha} \varepsilon x)$$

Since $(1-t)^{-1/\alpha} > 1$, we have $P(-X_1 > -(1-t)^{-\alpha} \varepsilon x) \geq P(-X_1 > -\varepsilon x)$. It follows from (3) and (4) that

$$\begin{aligned} \mathbb{P}\{Y_1 > (1 - \varepsilon)x\} &\geq \int_0^1 \mathbb{P}\{-X_1 > -\varepsilon x\} \Gamma(dt) \\ &= \mathbb{P}\{-X_1 > -\varepsilon x\} \mathbb{P}\{\sup_{0 \leq v \leq 1} Y_v > x\} \end{aligned} \quad (5)$$

Note that $\{X_u - X_1 : 0 \leq u \leq 1\} \stackrel{d}{=} \{-X_{1-u} : 0 \leq u \leq 1\}$. Therefore, Y_1 and $-I_1$ being the supremum of these two collections have the same distribution. By this fact and the definition of M_1^- , the inequality in (5) becomes

$$\mathbb{P}\{-I_1 > (1 - \varepsilon)x\} \geq \mathbb{P}\{-X_1 > -\varepsilon x\} \mathbb{P}\{M_1^- > x\}. \quad (6)$$

Observing that $P(-X_1 > -\varepsilon x) \rightarrow 1$ as $x \rightarrow \infty$, we get

$$\limsup_{x \rightarrow \infty} P(M_1^- > x)x^\alpha \leq \lim_{x \rightarrow \infty} P(-I_1 > (1 - \varepsilon)x)x^\alpha = (1 - \varepsilon)^{-\alpha} k$$

from (6) and Bertoin (2007, Prop.VIII.4), for every $\varepsilon > 0$. Combining this with (2), we have the result for $t = 1$ as ε can be made arbitrarily small. For $t > 0$, self-similarity yields $M_t^- \stackrel{d}{=} t^{1/\alpha} M_1^-$ to complete the proof. \square

3. Maximum Loss of Spectrally Negative Lévy Processes

In this section, we consider a spectrally negative Lévy process X which is not the negative of a subordinator or a deterministic drift. We study the distribution of maximum loss up to the passage time above a given level first. Then, we display the distribution up to a random independent exponential time.

The passage time above a level $x \geq 0$ is defined by

$$\tau_x = \inf\{t \geq 0 : X_t > x\} \quad (7)$$

which is also the right-continuous inverse of S by definition. Since S serves as a local time for the reflected process $S - X$ for spectrally negative Lévy process X , it follows that we can express its excursions from its maximum using τ . For brevity, we will use the notation τ instead of L^{-1} , which is the conventional notation for the right inverse of the local time. It is well known that τ is a subordinator (killed at an exponential time if X drifts to $-\infty$); see e.g. Bertoin (2007, Thm.VII.1). The points of discontinuity of $\tau = \{\tau_x : x \geq 0\}$ indicate the start of excursions ε_x of $S - X$, or excursions of X from its previous maximum, defined by

$$\varepsilon_x := \{X_{\tau_x} - X_{\tau_{x-}+s}, 0 < s \leq \tau_x - \tau_{x-}\}.$$

for $x > 0$ such that $\tau_x - \tau_{x-} > 0$ (Kuznetov et al., 2013). Indeed, $\tau_{x-} = \lim_{y \rightarrow x-} \tau_y$. If $\tau_x - \tau_{x-} = 0$, take $\varepsilon_x = \partial$ for some cemetery state ∂ . Then, $\{(x, \varepsilon_x) : x \geq 0, \varepsilon_x \neq \partial\}$ is a Poisson point process with values in $(0, \infty) \times \mathcal{E}$

where \mathcal{E} is the space of real valued right continuous left limited paths killed at the first hitting time of $(-\infty, 0]$, endowed with the σ -algebra generated by coordinates, and with mean measure

$$\nu(dx, d\varepsilon) = dx n(d\varepsilon)$$

where ε denotes a generic excursion due to homogeneity in space. The lifetime of an excursion is given by $\zeta := \inf \{s > 0 : \varepsilon(s) \leq 0\}$ and let

$$\bar{\varepsilon} := \sup \{\varepsilon(s) : s \leq \zeta\} .$$

When $\tau_x < \infty$, $X_{\tau_x} = x$ since spectrally negative Lévy process X creeps upwards having no positive jumps (Kyprianou, 2014).

For a spectrally negative Lévy process, there exists a unique continuous increasing function $W : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, called the *scale function*, such that the probability that X makes its first exit from an interval $[-x, y]$, $x, y > 0$, at y is

$$\mathbb{P}\{I_{\tau_y} \geq -x\} = \frac{W(x)}{W(x+y)} \quad (8)$$

for every $x, y > 0$, and the Laplace transform of W is given by

$$\int_0^\infty e^{-\lambda x} W(x) dx = \frac{1}{\psi(\lambda)}$$

see e.g. Bertoin (2007, Thm.VII.8). The scale function W is related to excursions by

$$\frac{W(x)}{W(y)} = \exp \left\{ - \int_x^y dt n(\bar{\varepsilon} > t) \right\}$$

for $y > x > 0$, Kyprianou (2014, Lem.8.2). It follows that

$$n(\bar{\varepsilon} > t) = \frac{W'_+(t)}{W(t)} \quad (9)$$

where W'_+ denotes the right derivative of W .

3.1. Maximum loss and maximum gain up to a first passage time

We find the joint distribution of maximum loss and maximum gain until the time that the process passes above a level $\beta > 0$, that is, till τ_β , as defined in (7). They are denoted by

$$M_{\tau_\beta}^- := \sup_{0 \leq u \leq v \leq \tau_\beta} (X_u - X_v), \quad M_{\tau_\beta}^+ = \sup_{0 \leq u \leq v \leq \tau_\beta} (X_v - X_u)$$

In particular, the marginal distribution of the maximum loss is obtained, consistently with the observation

$$M_{\tau_\beta}^- = \sup_{a < \beta} \bar{\varepsilon}_a .$$

An analogous expression is valid for the marginal distribution of maximum gain in terms of excursions of X from its minimum.

The event that X makes its first exit from an interval $[-x, y]$, $x, y > 0$, at the upper boundary point y can be written as $\{I_{\tau_y} \geq -x\}$ and will be used to find the joint distribution of maximum loss and maximum gain in this section. It follows as a corollary to the following theorem.

Theorem 2. *Consider a spectrally negative Lévy process X with scale function W . Then, for $u > 0, \alpha > 0, \beta > 0$, we have*

$$\mathbb{P}\{M_{\tau_\beta}^- \leq u, I_{\tau_\beta} \geq -\alpha\} = \begin{cases} e^{-\beta W'_+(u)/W(u)} & u \leq \alpha \\ \frac{W(\alpha)}{W(u)} e^{-(\beta + \alpha - u) W'_+(u)/W(u)} & \alpha \leq u \leq \alpha + \beta \\ \frac{W(\alpha)}{W(\alpha + \beta)} & \alpha + \beta \leq u \end{cases}$$

In particular, $\mathbb{P}\{M_{\tau_\beta}^- \leq u\} = e^{-\beta W'_+(u)/W(u)}$.

Proof: For $u > 0, \alpha > 0, \beta > 0$, we have

$$\{M_{\tau_\beta}^- \leq u\} = \{\forall a \in (0, \beta), \bar{\varepsilon}_a \leq u, \epsilon_a \neq \partial\}$$

Now, consider the sets $A = \{(a, \varepsilon_a) : 0 < a < \beta, \bar{\varepsilon}_a > u\}$ $B = \{(a, \varepsilon_a) : 0 < \alpha < \beta, \bar{\varepsilon}_a > a + \alpha\}$. Note that

$$\mathbb{P}\{M_{\tau_\beta}^- \leq u, I_{\tau_\beta} \geq -\alpha\} = \mathbb{P}\{N(A \cup B) = 0\}$$

Now, consider this probability in the following cases

- i. if $u \leq \alpha$, then $\{M_{\tau_\beta}^- \leq u\}$ is covered by $\{I_{\tau_\beta} \geq -\alpha\}$ and

$$\begin{aligned} \mathbb{P}\{M_{\tau_\beta}^- \leq u, I_{\tau_\beta} \geq -\alpha\} &= \mathbb{P}\{M_{\tau_\beta}^- \leq u\} = \mathbb{P}\{N(A) = 0\} \\ &= \exp(-\nu(A)) = e^{-\int_A da n(d\varepsilon)} = e^{-\beta n(\bar{\varepsilon} > u)} = e^{-\beta W'_+(u)/W(u)} \end{aligned}$$

by (9).

- ii. if $u \geq \alpha + \beta$ then $\{I_{\tau_\beta} \geq -\alpha\}$ is covered by $\{M_{\tau_\beta}^- \leq u\}$ and we have

$$\mathbb{P}\{M_{\tau_\beta}^- \leq u, I_{\tau_\beta} \geq -\alpha\} = \mathbb{P}\{I_{\tau_\beta} \geq -\alpha\} = \frac{W(\alpha)}{W(\alpha + \beta)}$$

iii. if $\alpha \leq u \leq \alpha + \beta$, then we have

$$\begin{aligned} & \mathbb{P}\{M_{\tau_\beta}^- \leq u, I_{\tau_\beta} \geq -\alpha\} \\ &= \mathbb{P}\{M_{\tau_\beta}^- \leq u, I_{\tau_\beta} \geq -\alpha \mid I_{\tau_{u-\alpha}} \geq -\alpha\} \mathbb{P}\{I_{\tau_{u-\alpha}} \geq -\alpha\} \\ &= \mathbb{P}_{u-\alpha}\{M_{\tau_\beta}^- \leq u, I_{\tau_{\beta-(u-\alpha)}} \geq -u\} \mathbb{P}\{I_{\tau_{u-\alpha}} \geq -\alpha\} \end{aligned}$$

where \mathbb{P}_x denotes the probability law of X starting from $x \in \mathbb{R}$ and we have used the strong Markov property. Similar arguments to those in i) yield

$$\begin{aligned} & \mathbb{P}_{u-\alpha}\{M_{\tau_\beta}^- \leq u, I_{\tau_{\beta-(u-\alpha)}} \geq -u\} \\ &= \mathbb{P}_{u-\alpha}\{M_{\tau_\beta}^- \leq u\} = e^{-(\beta+\alpha-u)n(\bar{\varepsilon}>u)} = e^{-(\beta+\alpha-u)W'_+(u)/W(u)} \end{aligned}$$

where the last equality is due to (9). Since $\mathbb{P}\{I_{\tau_{u-\alpha}} \geq -\alpha\} = W(\alpha)/W(u)$, the result follows.

When $\alpha \rightarrow \infty$, the marginal distribution follows. \square

The joint distribution of the maximum loss and maximum gain appears as in Salminen and Valois (2007) in view of the observation that $M_{\tau_\beta}^+ > \beta$ and for $\alpha > 0$

$$\{M_{\tau_\beta}^+ - \beta \leq \alpha\} = \{I_{\tau_\beta} \geq -\alpha\}.$$

We put $v = \alpha + \beta$ in Theorem 2 and get the following corollary.

Corollary 1. *For $v \geq \beta$ and $u \geq 0$, we have*

$$\mathbb{P}\{M_{\tau_\beta}^- \leq u, M_{\tau_\beta}^+ \leq v\} = \begin{cases} e^{-\beta n(\bar{\varepsilon}>u)} & u \leq v - \beta \\ \frac{W(v-\beta)}{W(u)} e^{-(v-u)n(\bar{\varepsilon}>u)} & v - \beta \leq u \leq v \\ \frac{W(v-\beta)}{W(v)} & v \leq u \end{cases}$$

3.2. Maximum loss and maximum gain up to an exponential time

Let the right inverse of ψ be denoted with ϕ , which is given by

$$\phi(\gamma) = \sup\{\lambda \geq 0 : \psi(\lambda) = \gamma\}$$

for $\gamma > 0$. In addition to the scale function W , there exist a family of increasing functions $W^{(q)} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ which satisfy

$$\int_0^\infty e^{-\lambda x} W^{(q)}(x) dx = \frac{1}{\psi(\lambda) - q},$$

for $q \geq 0$ and $\lambda > \phi(q)$, and also functions $Z^{(q)}$ defined on \mathbb{R} by

$$Z^{(q)}(x) = 1 + q \int_0^x W^{(q)}(y) dy . \quad (10)$$

The properties of the so-called q -scale functions $W^{(q)}$, and $Z^{(q)}$ can be gathered from Kyprianou (2014, Thm.8.1). Note that $W^{(0)} \equiv W$ above. Let $W_+^{(q)'}(x)$ denote the right derivative of $W^{(x)}$ at a .

Proposition 1. *For maximum loss and maximum gain up to an independent exponential time T with parameter $\gamma > 0$*

$$\begin{aligned} \mathbb{P}\{M_T^- > a\} &= Z^{(\gamma)}(a) - \gamma [W^{(\gamma)}(a)]^2 / W_+^{(\gamma)'}(a) \\ \mathbb{P}\{M_T^+ > a\} &= \frac{Z^{(\gamma)}(0)}{Z^{(\gamma)}(a)} \end{aligned}$$

for $a > 0$.

Proof: For the loss process, the first time to pass the level a is defined by

$$\bar{\sigma}_a = \inf\{v \geq 0 : S_v - X_v > a\}$$

The Laplace transform of $\bar{\sigma}_a$ has been found in Avram et al. (2004, Thm.1) as

$$\mathbb{E}(e^{-\gamma \bar{\sigma}_a}) = Z^{(\gamma)}(a) - \gamma [W^{(\gamma)}(a)]^2 / W_+^{(\gamma)'}(a) .$$

Clearly, we have $\mathbb{P}\{M_T^- > a\} = \mathbb{E}(e^{-\gamma \bar{\sigma}_a})$. Similarly, we can use the formula for $\mathbb{E}(e^{-\gamma \underline{\sigma}_a})$ from Pistorius (2004, Prop.2) to get the distribution of M_T^+ , where $\underline{\sigma}_a = \inf\{v \geq 0 : X_v - I_v > a\}$. \square

4. Infimum and Supremum of a Spectrally Negative Lévy Process

This section provides results that can be used towards finding the joint distribution of maximum loss and maximum gain up to an independent exponential time T for spectrally negative Lévy processes. The joint distribution has been derived for Brownian motion without drift in Salminen and Valois (2007). Through path decomposition at the times of the last supremum, and the last infimum before T , we find the distributions of the pre-supremum process and the post-infimum process. We provide the joint distribution of the supremum and the infimum drawing upon previous results. This can be used to derive the joint distribution of the maximum loss and maximum gain if the path can be decomposed in more detail than the present work.

4.1. Pre-supremum and post-infimum processes before an exponential time

In this part, we do a path decomposition of the Lévy process to characterize the distributions pre-supremum and post-infimum before an independent exponential time T with parameter $\gamma > 0$. Let X be a spectrally negative Lévy process, which is not the negative of a subordinator or a deterministic drift, as before. We define the first passage time below x as

$$\tau_x^- = \inf\{t \geq 0 : X_t < x\} \quad (11)$$

Let

$$H_S := \sup\{t < T : X_t = S_t\} \quad \text{and} \quad H_I := \sup\{t < T : X_t = I_t\}.$$

Then, we have the following characterization of pre- H_S process $\{X_u : 0 \leq u \leq H_S\}$ and post- H_I process $\{X_{H_I+u} - S_T : u \leq T - H_I\}$.

Theorem 3. *The pre- H_S process and the post- H_S process are independent. Respectively, the pre- H_I process and the post- H_I process are independent. Given $S_T = b$, the pre- H_S process is a spectrally negative Lévy process with Laplace exponent*

$$\bar{\psi}(\lambda) = \psi(\lambda + \phi(\gamma)) - \gamma$$

for $\lambda \geq -\phi(\lambda)$, killed at the first passage time above b . Given $I_T = a$, the post- H_I process is h -transform of the original spectrally negative Lévy process that started at a and killed at time $T \wedge \tau_a^-$ with

$$h(x) = \frac{\gamma W^\gamma(x-a)}{\phi(\gamma)} - \gamma \int_0^{x-a} W^\gamma(z) dz$$

for $x > b$.

Proof: Bertoin (2007, Lem.VI.6) gives the independence by noting that $X_{H_S-} = S_T$ if 0 is a regular point for $S - X$, equivalently, 0 is regular for $(-\infty, 0)$, which in turn is equivalent to X being of unbounded variation since X is spectrally negative, and $X_{H_S} = S_T$ if 0 is irregular for $S - X$. Using similar arguments, also as in Salminen and Valois (2007, Thm.3.2), and the excursions of X from its maximum, we get

$$\begin{aligned} & \mathbb{E} [F_1(X_u : u \leq H_S) F_2(X_{H_S+u} - S_T : u \leq T - H_S)] \\ &= \mathbb{E} \left[\int_0^\infty db F_1(X_u : u \leq \tau_b) e^{-\gamma \tau_b} \right] \mathbb{E} \left[\int_{\mathcal{E}} n(d\varepsilon) F_2(\varepsilon(u) : u \leq T) 1_{\{T < \zeta\}}(\varepsilon) \right] \end{aligned} \quad (12)$$

for bounded measurable functionals F_1 and F_2 . Note that we have used the fact that T is an exponential random variable with parameter $\gamma > 0$.

For the law of the pre- H_S process, we observe that

$$\begin{aligned}
& \mathbb{E}[F_1(X_u : u \leq \tau_b) e^{-\gamma \tau_b}] \\
&= \int \mathbb{P}(d\omega) F_1(X_u(\omega) : u \leq \tau_b(\omega)) e^{-\gamma \tau_b(\omega)} \\
&= \int \mathbb{P}^{\phi(\gamma)}(d\omega) e^{-\phi(\gamma)X_{\tau_b(\omega)}(\omega) + \psi(\phi(\gamma))\tau_b(\omega)} F_1(X_u(\omega) : u \leq \tau_b(\omega)) e^{-\gamma \tau_b(\omega)} \\
&= \mathbb{E}^{\phi(\gamma)}[F_1(X_u : u \leq \tau_b)] e^{-\phi(\gamma)b}
\end{aligned}$$

by change of measure with $c = \phi(\gamma)$ as given in Kyprianou (2013, pg.233), where \mathbb{P}^c is the law of another spectrally negative Lévy process with Laplace exponent $\bar{\psi}(\lambda) = \psi(\lambda + c) - \psi(c)$. From (12), we get

$$\begin{aligned}
& \mathbb{E}[F_1(X_u : u \leq H_S) F_2(X_{H_S+u} - S_T : u \leq T - H_S)] \tag{13} \\
&= \int_0^\infty db \mathbb{E}[F_1(X_u : u \leq \tau_b)] e^{-\phi(\gamma)b} \int_{\mathcal{E}} n(d\varepsilon) \mathbb{E}[F_2(\varepsilon(u) : u \leq T) 1_{\{T < \zeta\}}(\varepsilon)] \\
&= \frac{1}{\phi(\gamma)} \int_0^\infty db \mathbb{E}[F_1(X_u : u \leq \tau_b)] f_{S_T}(b) \int_{\mathcal{E}} n(d\varepsilon) \mathbb{E}[F_2(\varepsilon(u) : u \leq T) 1_{\{T < \zeta\}}(\varepsilon)]
\end{aligned}$$

where we wrote f_{S_T} for the probability density function of S_T , which has exponential distribution with parameter $\phi(\gamma)$ (Kyprianou, 2014, pg.233). The assertion about the conditional distribution of the pre-process given $S_T = b$ follows from (13).

The independence of pre- H_I and post- H_I process follows from Bertoin (2007, Lem.VI.6) in view of duality between $X - I$ and $\hat{S} - \hat{X}$ and with the analysis as in (12) with the excursion measure for $\hat{S} - \hat{X}$. Now, consider the distribution of post- H_I process characterized by

$$\mathbb{E} \left[\int_{\mathcal{E}} \hat{n}(d\varepsilon) F(\varepsilon(u) : u \leq T) 1_{\{T < \zeta\}}(\varepsilon) \right]$$

for measurable bounded functions F of excursions above the infimum. This expectation is equal to

$$\gamma \int_0^\infty dt e^{-\gamma t} \int_{\mathcal{E}} \hat{n}(d\varepsilon) F(\varepsilon(u) : u \leq t) 1_{\{t < \zeta\}}(\varepsilon)$$

which can be written as

$$\gamma W(0+) \int_0^\infty dt e^{-\gamma t} \mathbb{E}^\uparrow[F(X_u : u \leq t) W(X_t)^{-1}]$$

from Bertoin (2007, Prop.VII.15), where $F(X_u : u \leq t)$ is measurable with respect to the underlying filtration \mathcal{F}_t . In particular, we have

$$\begin{aligned}
P_T f(0) &\propto \gamma W(0+) \int_0^\infty dt e^{-\gamma t} \int_0^\infty \mathbb{P}_0^\uparrow(X_t \in dy) W(y)^{-1} f(y) \\
&\propto \int_0^\infty \int_0^\infty e^{-\gamma t} \mathbb{P}_0(\tau_y^+ \in dt) f(y) dy \\
&= \int_0^\infty f(y) dy \int_0^\infty e^{-\gamma t} \mathbb{P}_0(\tau_y^+ \in dt) = \int_0^\infty f(y) e^{-\phi(\gamma)y} dy
\end{aligned}$$

from Bertoin (2007,pg.204,190). Therefore, the value of the post- H_I process at time T is the sum of I_T and an exponential random variable with parameter $\phi(\gamma)$. At an intermediate time $t < T$, the post- H_I process is a spectrally negative Lévy process which stays above I_T . The probability law of this process is written by

$$\begin{aligned}
\mathbb{P}_x\{X_t \in dy, t < T \mid T < \tau_0^-\} &= \mathbb{P}_x\{X_t \in dy, t < T, t < \tau_0^- \mid T < \tau_0^-\} \quad (14) \\
&= \frac{\mathbb{P}_x\{X_t \in dy, t < T \wedge \tau_0^-, T < \tau_0^-\}}{\mathbb{P}_x\{T < \tau_0^-\}} \\
&= \frac{\mathbb{P}_x\{X_t \in dy, t < T \wedge \tau_0^-, T \circ \theta_t < \tau_0^- \circ \theta_t\}}{\mathbb{P}_x\{T < \tau_0^-\}} \\
&= \frac{\mathbb{P}_y\{T < \tau_0^-\}}{\mathbb{P}_x\{T < \tau_0^-\}} \mathbb{P}_x\{X_t \in dy, t < T \wedge \tau_0^-\}
\end{aligned}$$

by Markov property, where θ is the shift operator. It follows that the law of the post- H_I process is the h -transform of a spectrally negative Lévy process killed at time $T \wedge \tau_0^-$ with

$$h(x) = \mathbb{P}_x\{T < \tau_0^-\}.$$

The function h can be evaluated as

$$\begin{aligned}
h(x) &= \mathbb{E}_x[\mathbb{P}_x\{T < \tau_0^- \mid \tau_0^-\}] = \mathbb{E}_x[1 - e^{-\gamma\tau_0^-} 1_{\{\tau_0^- < \infty\}}] \\
&= 1 - \mathbb{E}_x[e^{-\gamma\tau_0^-} 1_{\{\tau_0^- < \infty\}}] \\
&= 1 - Z^\gamma(x) + \frac{\gamma}{\phi(\gamma)} W^\gamma(x) \\
&= \frac{\gamma W^\gamma(x)}{\phi(\gamma)} - \gamma \int_0^x W^\gamma(z) dz
\end{aligned}$$

by Kyprianou (2014, Thm.8.1) and (10). □

Note that the post- H_I process can also be viewed as a transform of the law of the Lévy process conditioned to stay positive (Bertoin, 2007, pg.198) by rearranging (14) as

$$\begin{aligned}
& \mathbb{P}_x\{X_t \in dy, t < T \mid T < \tau_0^-\} \\
&= \frac{h(y)}{h(x)} \mathbb{P}_x\{X_t \in dy, t < \tau_0^-, t < T\} \\
&= \frac{h(y)/W(y)}{h(x)/W(x)} e^{-\gamma t} \frac{W(y)}{W(x)} \mathbb{P}_x\{X_t \in dy, t < \tau_0^-\} \\
&=: \frac{\tilde{h}(y)}{\tilde{h}(x)} e^{-\gamma t} \mathbb{P}_x^\uparrow\{X_t \in dy\} = \frac{\tilde{h}(y)}{\tilde{h}(x)} \mathbb{P}_x^\uparrow\{X_t \in dy, t < T\}
\end{aligned}$$

where $\tilde{h} := h/W$ and \mathbb{P}_x^\uparrow denotes the law of the Lévy process started at x and conditioned to stay positive. As a result, the post- H_I process is \tilde{h} -transform of the law of the Lévy process conditioned to stay positive and killed at an exponential time.

As a special case, $h(x) = 1 - e^{-x\sqrt{2\gamma}}$ for standard Brownian motion as given in Salminen and Valois (2007, Thm.2.7). This follows because $\tau_0^- = \tau_0^+$ by symmetry and $\mathbb{E}_x e^{-\gamma\tau_0^+} = e^{-x\sqrt{2\gamma}}$.

4.2. Joint distribution of the supremum and the infimum

In this part, we find the joint distribution of supremum and infimum of a spectrally negative Lévy process X , which is taken up to an independent exponential time T with parameter γ . This distribution is characterized by the Laplace transform of $\tau_x \wedge \tau_0^-$ where τ_x is the first passage time above a level $x \geq 0$ as in (7) and τ_x^- is below x as defined in (11).

The process X moves continuously upwards therefore it will hit the level x . However, depending on its components, the process hits level 0 or jumps below it.

Proposition 2. *Let, X , be a spectrally negative Lévy process and let T be an exponentially distributed random variable with parameter γ , independent of X . For $a < 0 < b$, the joint distribution of I_T and S_T is given by*

$$\mathbb{P}_0(a < I_T, S_T < b) = 1 - Z^{(\gamma)}(-a) + [Z^{(\gamma)}(b - a) - 1] \frac{W^{(\gamma)}(-a)}{W^{(\gamma)}(b - a)}.$$

Proof: For $a < 0 < b$, we have

$$\begin{aligned}
\mathbb{P}_0\{a < I_T, S_T < b\} &= \mathbb{P}_0\{T < \tau_a^- \wedge \tau_b\} \\
&= \mathbb{P}_0\{T < \tau_a^-, \tau_a^- < \tau_b\} + \mathbb{P}_0\{T < \tau_b, \tau_b < \tau_a^-\} \\
&= 1 - (\mathbb{P}_0\{T > \tau_a^-, \tau_a^- < \tau_b\} + \mathbb{P}_0\{T < \tau_b, \tau_b < \tau_a^-\}) \\
&= 1 - (\mathbb{E}_0[e^{-\gamma\tau_a^-} 1_{\{\tau_a^- < \tau_b\}}] + \mathbb{E}_0[e^{-\gamma\tau_b} 1_{\{\tau_b < \tau_a^-\}}]) \\
&= 1 - (\mathbb{E}_{-a}[e^{-\gamma\tau_0^-} 1_{\{\tau_0^- < \tau_b - a\}}] + \mathbb{E}_{-a}[e^{-\gamma\tau_b - a} 1_{\{\tau_b - a < \tau_0^-\}}])
\end{aligned}$$

The expectations in the above equation are known as two-sided exit from below and two sided-exit from above a level, respectively. For a spectrally negative Lévy process, two sided exit problems are well defined and the expressions for them are well known (Kuznetsov et al., 2013). After plugging these expressions, we find the joint distribution as

$$\begin{aligned}
P_0(a < I_T, S_T < b) & \tag{15} \\
&= 1 - \left[Z^{(\gamma)}(-a) - Z^{(\gamma)}(b-a) \frac{W^{(\gamma)}(-a)}{W^{(\gamma)}(b-a)} + \frac{W^{(\gamma)}(-a)}{W^{(\gamma)}(b-a)} \right] \\
&= 1 - Z^{(\gamma)}(-a) + [Z^{(\gamma)}(b-a) - 1] \frac{W^{(\gamma)}(-a)}{W^{(\gamma)}(b-a)}
\end{aligned}$$

In order to derive the joint distribution of the maximum loss and maximum gain, one could aim at decomposing the path of X in more detail than the present work, including the intermediate process between the infimum and the supremum. Then, conditioning on the infimum occurring before the supremum, or vice versa, and the joint distribution of the supremum and the infimum could be useful. For Brownian motion without drift, symmetry helps as shown in Salminen and Valois (2007). Since a spectrally negative Lévy process does not have this property in general, the joint distribution of M_T^- and M_T^+ remains as future work.

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